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A CONTRIBUTION TO THE INTEGRAL THEORY OF A TURBULENT JET ABOVE A POINT HEAT SOURCE OF CONSTANT STRENGTH

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Consideration has been given to the integral model of an unsteady vertical convective jet above a point heat source of constant strength. It has been shown that this problem is reduced to self-similar equations allowing the analytical solution. An algebraic invariant relating the parameters of the velocity and the temperature along the jet axis has been constructed. A comparison of the analytical solution and the existing experimental data on the propagation of the upper boundary of the convective jet has been made.

Introduction. Unsteady forced convective jets are associated mainly with technogenic heat sources whose strength changes with time depending on the technological process. The case of a constant source corresponds to the most typical situation of regular operation of an industrial object. Developing flames are observed above manifolds, electric power stations, and other industrial heat sources, being of undeniable practical interest in connection with problems of calculation and prediction of propagation of impurities.

Allowing for the fact that stratification effects virtually do not manifest themselves at low heights, the approximation of a neutral medium is quite suitable for description of convective jets in the lower 50 m of the atmosphere, which are adjacent to the underlying surface.

The first theoretical description of unsteady penetrative convection above a constant point heat source was performed in [1], where a developing flame was considered as a composition of two well-known integral models: the model of a steady-state conical jet and the model of a spherical vortex ring. An analogous approach was employed later in [2].

A fundamentally new integral model of an unsteady convective jet, which considers the flame as a conical surface with an unsteady cross section, was proposed in [3, 4].

A variant of the integral model of an unsteady floating jet, which employs the notion of a convective front [5], was realized in [6, 7]. Within the framework of this model, the unsteady convective jet is considered as a cone with a rising upper cross section. It is precisely this approach that has provided the basis for analytical study of a developing flame within the framework of the present work.

Problem of a Floating Jet above a Point Heat Source. We consider the problem of propagation of an axisymmetric unsteady convective jet in an adiabatic atmosphere above a point heat source. Let the z axis be directed in opposition to the free-fall acceleration g in the cylindrical coordinate system r, φ, z.

To describe the propagation of the jet we employ the Boussinesq convection equations [8] which hold true for both a gas and a liquid. Let $\overline{\Theta}_a$ = const be the static value of the potential temperature of dry air in a stationary atmosphere and Θ be the local potential temperature of the air.* Following [8], we introduce $\theta = (\Theta - \overline{\Theta}_a)/\overline{\Theta}_a$, i.e., the local dimensionless potential temperature.**

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* For an ideal gas the potential temperature Θ is determined by the relation $\Theta = T(p/p_n)^{-R_d/c_p}$, where the constant normal pressure of the gas on the underlying surface p_n is equal to approximately 1 atm.

** If an incompressible liquid with a constant background density ρ_{hom} is selected as the medium, then $\theta = -(\rho - \rho_{hom})/\rho_{hom}$.

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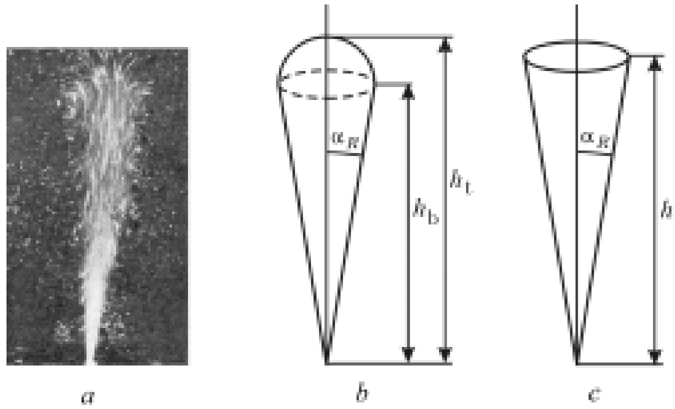

Fig. 1. Propagation of the unsteady jet according to [1] (a) and diagrammatic representations of the developing flame according to the models of [1, 2] (b) and to the present model (c).

We note that, under real conditions, the process of heating is localized in the comparatively small region near the underlying surface. This circumstance enables us to subsequently style the effect of heating by assigning the corresponding point heat source on the underlying surface. Flow in the axisymmetric unsteady convective jet will be considered in the approximation of a vertical boundary layer [9]:

$$\frac{\partial}{\partial t} w + \frac{1}{r}\frac{\partial}{\partial r} uwr + \frac{\partial}{\partial z} ww = g\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(\nu_w r \frac{\partial w}{\partial r}\right),$$

$$\frac{\partial}{\partial t}\theta + \frac{1}{r}\frac{\partial}{\partial r} u\theta r + \frac{\partial}{\partial z} w\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu_\theta r \frac{\partial \theta}{\partial r}\right), \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r} ur + \frac{\partial}{\partial z} w = 0 .$$

The system of equations (1) is considered in the region $V = \{0 \le r < \infty,\, 0 \le \varphi \le 2\pi,\, 0 \le z \le \infty\}$, where the infinite upper boundary of the region is identified with the height of the adiabatic atmosphere.

The state of the stationary atmosphere will be taken as the initial condition at t_0. Allowing for the fact that the medium is unperturbed at the upper and lateral boundaries of the region, we take the conditions of nonflow and of attenuation of flows. At the lower boundary of the region, we assign the strength of the steady-state point heat source and the zero pulse source:

$$\lim_{z\to 0}\,[w\cdot w\,(r, z, t)] = 0\,, \quad \lim_{z\to 0}\,[w\cdot\theta\,(r, z, t)] = \frac{1}{2\pi r} Q_1 \delta\,(r)\,, \tag{2}$$

where $Q_1 > 0$. Relations (1) and (2) form a closed system of equations.

Integral Model of a Convective Jet Above a Point Heat Source. The real pattern of development of a flame above a constant source according to the laboratory data of [1] is prescribed in Fig. 1a. In theoretical models [1, 2], the shape of the convective jet is approximated by a conical surface of constant slope α_R, with a height h_b and a spherical head part of radius $\alpha_R h_b$, so that the relation $h_t = (1 + \alpha_R)h_b$ holds true for the maximum height of the flame h_t (Fig. 1b). In the case of a stained flame the parameters $h_t = h_t(t)$ and $h_b = h_b(t)$ are easily fixed from experimental data.

In the present model, the convective jet is approximated by a cone of equivalent volume with a plane upper base (Fig. 1c). The height of the lateral surface of the cone $h = h(t)$ is assigned by the relation $h = (1 + \alpha_R/3)h_b$, which can be computed from the data of observations.

Thus, in accordance with the Prandtl hypothesis and the models of [6, 7], for the radius of the jet R above the point source we take the law of linear expansion

$$R(z,t)=\alpha_R z\,,\quad 0\le z\le h(t)\,,\tag{3}$$

where the value of α_R varies within 0.10–0.18 according to [2]. We do not consider motion in the region $z > h(t)$ within the framework of subsequent discussion.

The equation describing the propagation of the upper boundary of the conical convective jet in a neutral atmosphere above a constant heat source can be obtained based on the dimensional theory [1–3]

$$h(t)=\left\{\frac{1}{\lambda_0^2}\left(\frac{4}{3}\right)^2 gQ_1\right\}^{1/4} t^{3/4}\,,\tag{4}$$

where λ_0^2 is the undetermined numerical parameter.*

We employ the integral method of Karman and Pohlhausen for construction of the approximate solution of the system [9]. It is assumed that the unknown functions in the region of ascending motion $0 \le z \le h(t)$ are approximated by relations with separable variables, in which the ~ sign corresponds to the parameters on the jet axis:

$$\begin{gathered}
w(r,z,t)=\tilde{w}(z,t)\,f_w\left(\frac{r}{R}\right),\\
u(r,z,t)=-\frac{\partial\tilde{w}(z,t)}{\partial z}\,\frac{1}{r}\int_0^r rf_w\left(\frac{r}{R}\right)dr\,,\\
\theta(r,z,t)=\tilde{\theta}(z,t)\,f_\theta\left(\frac{r}{R}\right).
\end{gathered}\tag{5}$$

To compare to the existing models [3, 4] we employ the exponential approximations of the parameter profiles in accordance with the known experimental data [10]:

$$f_w(\xi)=\exp(-\lambda_w\xi^2)\,,\quad f_\theta(\xi)=\exp(-\lambda_\theta\xi^2)\,,\quad \xi=r/R\,,\tag{6}$$

where, from the data of [10], $\lambda_w/\lambda_\theta = 1.35$ and $\lambda_w = 96\alpha_R^2$.

Substituting (5) and (6) into Eqs. (1) and integrating the equations obtained over the cross-sectional area of the jet, we obtain

$$\begin{gathered}
\frac{\partial}{\partial t}\tilde{w}R^2+\frac{1}{2}\frac{\partial}{\partial z}\tilde{w}\tilde{w}R^2=\alpha_g g\tilde{\theta}R^2\,;\\
\frac{\partial}{\partial t}\tilde{\theta}R^2+\frac{1}{1+\alpha_g}\frac{\partial}{\partial z}\tilde{w}\tilde{\theta}R^2=0\,,
\end{gathered}\tag{7}$$

here $\alpha_g = \lambda_w/\lambda_\theta$ is a constant coefficient.

We should supplement Eqs. (7) with the boundary conditions

$$\lim_{z\to 0}\,[\tilde{w}\tilde{w}R^2(z,t)]=0\,,\quad \lim_{z\to 0}\,[\tilde{w}\tilde{\theta}R^2(z,t)]=\frac{1}{\pi}\frac{\lambda_w}{k^2}Q_1\,,\quad k^2=\frac{\alpha_g}{1+\alpha_g}\,.\tag{8}$$

To compute the unknown parameter λ_0^2 we employ the kinematic equation at the upper boundary of the thermal (air bubble). Following [5, 6], we introduce the characteristic velocity of movement of the convective front $\hat{w}(h,t)$ as a weighted average over the temperature:

* The factor 4/3 in (4) has been employed for consistency with the previous works of the authors.

$$\hat{w}(h,t)=\frac{2\pi\int_0^\infty w\theta r dr}{2\pi\int_0^\infty \theta r dr}=\tilde{w}(h,t)\frac{2\pi\int_0^\infty f_w(\xi)\,f_\theta(\xi)\,\xi d\xi}{2\pi\int_0^\infty f_\theta(\xi)\,\xi d\xi}=\frac{1}{1+\alpha_g}\tilde{w}(h,t)\,. \tag{9}$$

Assuming that the movement of the upper boundary is determined by the convective heat flux [5], we represent the kinematic condition for $z = h(t)$ in the form

$$\frac{dh}{dt}=c^{-2/3}\hat{w}(h,t)=c^{-2/3}(1+\alpha_g)^{-1}\tilde{w}(h,t)\,, \tag{10}$$

where c is the assigned constant. According to [6], the values $c = 1$ and $c^2 = k^2 = \alpha_g/(1+\alpha_g)$ are theoretically justified.

Self-Similar Solution of the Development of a Jet above a Constant Point Heat Source. Let $z_* = z/h(t)$ be the dimensionless parameter. For $0 < z < h(t)$ we will seek the self-similar solution (7) in the form

$$\begin{aligned}\tilde{w}(z,t)&=\frac{dh}{dt}w^*(z_*)\,,\\ \tilde{\theta}(z,t)&=\frac{1}{gh}\left(\frac{dh}{dt}\right)^2\theta^*(z_*)\,,\\ R(z,t)&=hR_*=h\alpha_R z_*\,.\end{aligned} \tag{11}$$

The substitution of (11) into system (7) leads to a system of ordinary differential equations when $0 < z_* < 1$:

$$\begin{aligned}&\frac{8}{3}w^*R_*^2-\frac{d}{dz_*}(w^*R_*^2z_*)+\frac{1}{2}\frac{d}{dz_*}w^*w^*R_*^2=\alpha_g\theta^*R_*^2\,,\\ &\frac{4}{3}\theta^*R_*^2-\frac{d}{dz_*}(\theta^*R_*^2z_*)+\frac{1}{1+\alpha_g}\frac{d}{dz_*}w^*\theta^*R_*^2=0\,,\\ &R_*=\alpha_R z_*\,.\end{aligned} \tag{12}$$

In accordance with (8), boundary conditions (12) have the form*

$$\begin{aligned}&\lim_{z_*\to 0}\left\{w^*\cdot w^*\cdot R_*^2\right\}=0\,,\\ &\lim_{z_*\to 0}\left\{w^*\cdot\theta^*\cdot R_*^2\right\}=S_*\,,\quad S_*=\frac{1}{\pi}\frac{\lambda_w}{k^2}\frac{gQ_1}{h\left(\frac{dh}{dt}\right)^3}=\frac{4}{3\pi}\frac{\lambda_w}{k^2}\lambda_0^2\,.\end{aligned} \tag{13}$$

The system of equations (12) allows the analytical solution

$$w^*(z_*)=\frac{1}{\alpha_R z_*}\left\{\frac{3}{2}\alpha_g\alpha_R S_* z_*^2\right\}^{1/3}\,,$$

* Analogous self-similar relations can also be obtained for heat sources whose strengths vary according to arbitrary power laws and for instantaneous and exponential sources (see [6]).

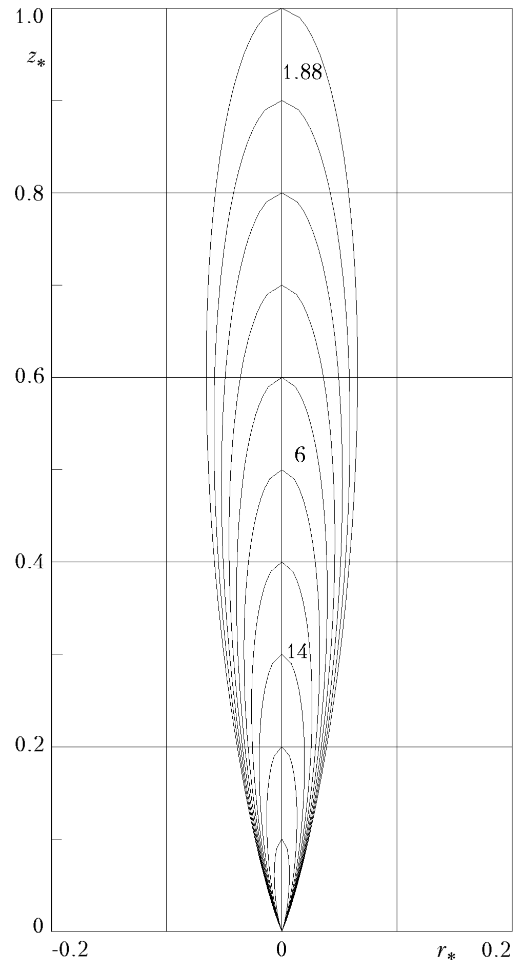

Fig. 2. Isolines of the field of the normalized dimensionless potential temperature $\theta_*(r_*, z_*)$ in self-similar dimensionless spatial variables r_* and z_*.

$$\theta^*(z_*) = \frac{S_*}{\alpha_R z_*}\left\{\frac{3}{2}\alpha_g \alpha_R S_* z_*^2\right\}^{-1/3}, \tag{14}$$

$$R_*(z_*) = \alpha_R z_*, \quad S_* = \frac{4}{3\pi}\frac{\lambda_w}{k^2}\lambda_0^2 .$$

The undetermined factor λ_0^2 can be computed from kinematic condition (10) and Eqs. (11) and (14). We have

$$\lambda_0^2 = \frac{\pi}{2}\frac{c^2}{\lambda_w}\alpha_R^2 (1+\alpha_g)^2 . \tag{15}$$

We have $\lambda_0^2 = 9.03 \cdot 10^{-2}$ when $c^2 = 1$, and $\lambda_0^2 = 5.15 \cdot 10^{-2}$ when $c^2 = k^2 = \alpha_g/(1+\alpha_g) = 0.57$. The parameter λ_0^2 computed agrees fairly well with $\lambda_0^2 = 7.02 \cdot 10^{-2}$ obtained from observations of the propagation of thermals in air [11].

The amplitudes of (14) supplemented with the profile relations (6) enable us to calculate the spatial field of the unsteady jet in self-similar variables $z_* = z/h$ and $r_* = r/h$. In particular, for the potential temperature we have $\theta_*(r_*, z_*) = \theta^*(z_*) \exp\left\{-71(r_*/z_*)^2\right\}$. The numerical calculations from the above formula for $\lambda_0^2 = 5.15 \cdot 10^{-2}$ and $\alpha_R = 0.12$ are presented in Fig. 2.

TABLE 1. Computed Values of the Parameter a

c^2	α_R				
	0.10	0.12	0.14	0.16	0.18
1	0.41	0.40	0.40	0.40	0.39
0.57	0.49	0.49	0.48	0.48	0.47

Comparison to Laboratory Experiments. For comparison to the data given in [1, 12], we pass from the height h to the height h_b observed (see Fig. 1b). Allowing for the fact that $h = (1 + \alpha_R/3)h_b$ and taking into account (11) and (14), we obtain $\tilde{w}(h)/\tilde{w}(h_b) = (1 + \alpha_R/3)^{1/3}$. Thus, the kinematic condition (10) takes the form

$$\frac{dh_b}{dt} = a\tilde{w}\,(h_b, t)\,, \quad a = c^{-2/3}\,(1 + \alpha_g)^{-1}\,(1 + \alpha_R/3)^{-4/3}\,. \tag{16}$$

The values of the parameter a computed from formula (16) are presented in Table 1 for assigned values of c.

The data obtained are in good quantitative agreement with the experimental values of $a = 0.49$ and $a = 0.47$ given in [1] and [2] respectively.

Algebraic Invariant of an Unsteady Jet. We consider the ratio of the kinetic and potential energy along the axis of an unsteady jet. Then, in accordance with (11) and (14), we obtain

$$\frac{1}{2}\frac{\tilde{w}^2}{g\tilde{\theta}z} = \frac{1}{2}\frac{(w^*)^2}{\theta^* z_*} = \frac{3}{4}\,\alpha_g = 1.01\,. \tag{17}$$

Relation (17) represents the local energy invariant establishing the algebraic relationship of the amplitude parameters along the axis of the unsteady convective jet. Formula (17) can be interpreted in the same manner as the law of uniform energy distribution in degrees of freedom along the jet axis.

The presence of such invariants is very typical of convection problems. The existence of the invariants was pointed to for the first time by Scorer [12] in experimental investigation of supernatant thermals. The corresponding invariants for a thermal, a steady-state forced jet, and a steady-state spontaneous jet have been give in [6, 13, 14].

We consider the problem of energetics of the unsteady convective jet

$$\begin{aligned} E_k &= 2\pi\int_0^\infty \frac{w^2}{2}\,rdr = \pi R^2\left(\frac{dh}{dt}\right)^2 (w^*)^2 \int_0^\infty f_w^2\,(\xi)\,\xi d\xi = \pi R^2\left(\frac{dh}{dt}\right)^2 (w^*)^2\,\frac{1}{4\lambda_w}\,, \\ E_p &= 2\pi\int_0^\infty g\theta z r dr = 2\pi R^2\left(\frac{dh}{dt}\right)^2 (\theta^* z_*)\int_0^\infty f_\theta\,(\xi)\,\xi d\xi = \pi R^2\left(\frac{dh}{dt}\right)^2 (\theta^* z_*)\,\frac{1}{\lambda_\theta}\,. \end{aligned} \tag{18}$$

With account for (17), we compute the ratio

$$\frac{E_k}{E_p} = \frac{1}{4}\,\frac{\lambda_\theta}{\lambda_w}\,\frac{(w^*)^2}{(\theta^* z_*)} = \frac{3}{8}\,\alpha_g\,\frac{\lambda_\theta}{\lambda_w} = \frac{3}{8}\,. \tag{19}$$

Expression (19) means that the same law of energy distribution in degrees of freedom is obeyed at any cross section of the convective jet.

CONCLUSIONS

The proposed self-similar solution of the problem of an unsteady convective jet above a constant-strength point source corresponds rather well to the existing experimental data. The analytical solution of the problem proves the existence of an algebraic invariant relating the amplitude characteristics along the jet axis.

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NOTATION

u and w, velocity components along the r and z axes respectively, m/sec; ν_w and ν_{θ_2}, coefficients of turbulent exchange for the vertical velocity and the dimensionless potential temperature, m^2/sec; Q_1, constant strength of the point heat source, m^3/sec; $\delta(r)$, Dirac delta function; $\tilde{w}$ and $\bar{\theta}$, vertical velocity and dimensionless potential temperature on the axis of the unsteady jet, m/sec; h, height of the upper boundary of the jet, m; f_w and f_θ, dimensionless horizontal profiles of the vertical velocity and the potential temperature; λ_w and λ_θ, numerical parameters characterizing the dimensionless horizontal profiles of the vertical velocity and the potential temperature; w^*, R_*, and S^*, normalized dimensionless functions of the vertical velocity, the radius, and the strength of the heat source; t, time, sec; t_0, initial instant of time, sec; r, φ, z, cylindrical coordinate system, g, free-fall acceleration, m/sec^2; $\bar{\Theta}_a$, static value of the potential temperature of dry air in a stationary atmosphere; Θ, local potential temperature of the air; T, potential temperature of the gas, K; p and p_n, local pressure and constant normal pressure of the gas on the underlying surface, Pa; R_d, gas constant of the dry air; c_p, specific heat capacity at constant pressure; θ, local dimensionless pulsation of the potential temperature, dimensionless quantity; ρ_{hom}, constant background density of a homogeneous liquid; ρ, local value of the liquid density; R, radius of the jet, m; α_R, coefficient of angular expansion of the jet radius; h_b, height of the lateral surface of the jet cone, m; h_t, height of the center the jet, m; z_* and r_*, normalized vertical and horizontal coordinates; θ^* and θ_*, normalized potential temperature on the jet axis and at arbitrary points r_* and z_*; E_k and E_p, kinetic and potential energies in the assigned cross section of the jet, m^4/sec^2. Subscripts and superscripts: 0, initial; 1, steady-state; a, adiabatic; n, normal; hom, homogeneous; d, dry; b, lower; t, upper; k, kinetic; p, potential; *, dimensionless.

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